

# **WeibPar**

## ***Time Independent Theory Guide***

Fast Fracture Weibull Parameter Estimation

***CONNECTICUT RESERVE TECHNOLOGIES, INC.***

# A

## THEORY: PARAMETER ESTIMATION

### **A.1 INTRODUCTION**

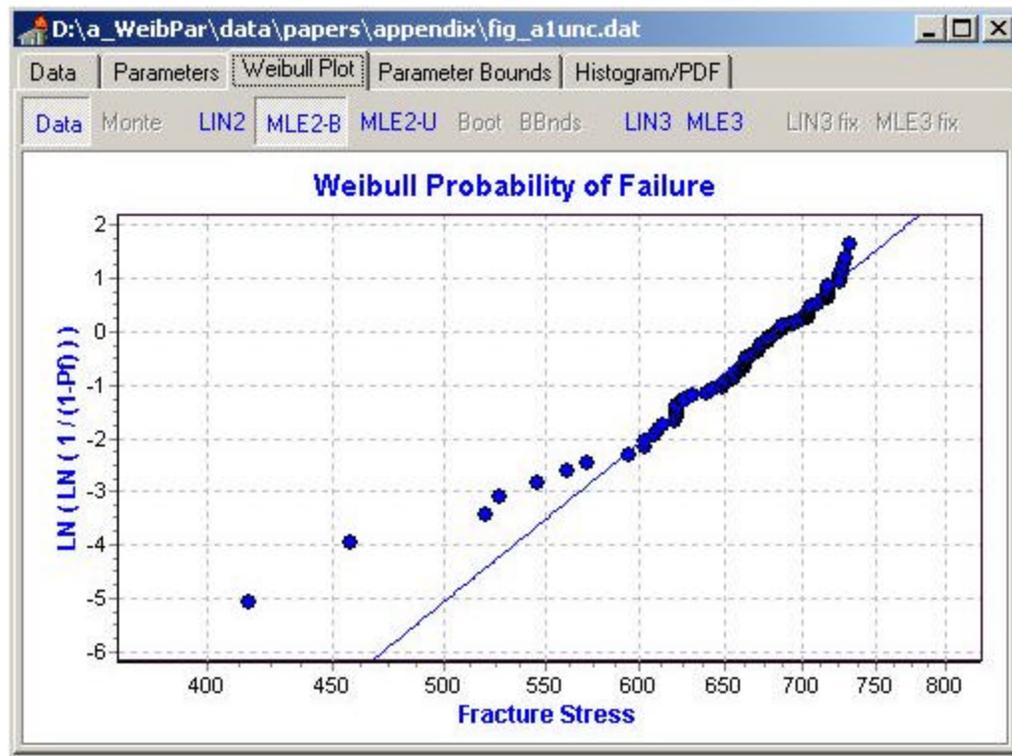
The WeibPar program that accompanies this discussion produces estimated values for unknown Weibull distribution parameters based on observations recorded in strength to failure tests. The program and estimation methods are applicable to ceramic materials (monolithic or composite) that do not exhibit any appreciable bilinear or nonlinear deformation behavior. If the techniques are applied to failure data from composite materials then the composite must contain a uniformly distributed second phase (e.g., whiskers, short fibers, etc.) such that the composite is effectively homogeneous. In essence the material must behave in a linear elastic, brittle fashion if the user wishes to analyze the failure data by the methods that follow.

Strength measurements are taken for one of two reasons: either for a comparison of the relative quality of two materials, or for the prediction of the failure probability for a structural component. The analytical details provided here allow for either. In order to obtain point estimates of the unknown Weibull distribution parameters, well-defined functions are utilized that incorporate the failure data and specimen geometry. These functions are referred to as estimators. It is desirable that an estimator be consistent and efficient. In addition, the estimator should produce unique, unbiased estimates of the distribution parameters. Different types of estimators exist, including moment estimators, least-squares estimators, and maximum likelihood estimators. This discussion initially focuses on maximum likelihood estimators (MLE) due to the efficiency and the ease of application when censored failure populations are encountered. The likelihood estimators are used to compute parameters from failure populations characterized by a two parameter Weibull distribution. Alternatively, non-linear regression estimators (discussed later) are utilized to calculate unknown distribution parameters for a three parameter Weibull distribution. Basically, this entire discussion provides a theoretical background for the calculation of parameter estimates that take place within the WeibPar program.

Many factors affect the estimates of the distribution parameters. The total number of test specimens plays a significant role. Initially, the uncertainty associated with parameter estimates decreases significantly as the number of test specimens increases. However a point of diminishing returns is reached when the cost of performing additional strength tests may not be justified. This suggests that a practical number of strength tests should be performed to obtain a desired level of confidence associated with a parameter estimate. This point can not be overemphasized. However, quite often 30 specimens (a widely cited rule-of-thumb) is deemed a sufficient quantity of test specimens when estimating Weibull parameters. One should immediately ask why 29 specimens would not suffice. Or more importantly, why is 30 specimens sufficient? The answer to this is addressed in a later section where the details of computing confidence bounds for the maximum likelihood estimates (these bounds are directly relate to the precision of the estimate) are presented. Confidence bounds for the non-linear regression estimators are not available for reasons cited in reference [1].

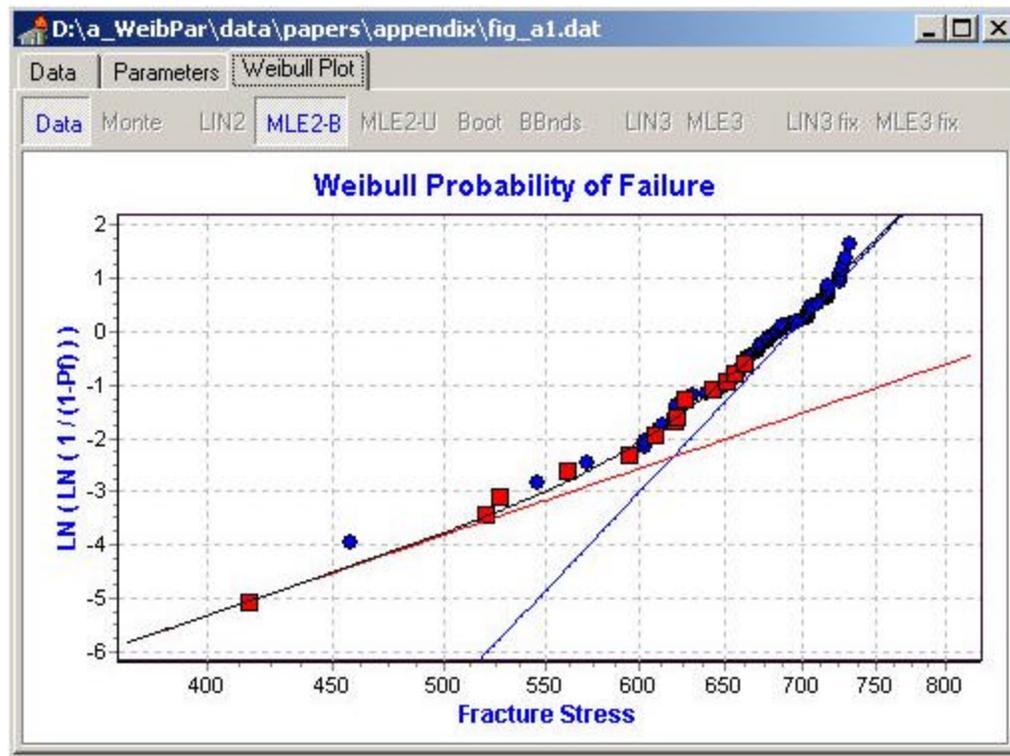
Tensile and flexural specimens are the most commonly used test configurations for ceramic materials. However, most ceramic material systems exhibit a decreasing trend in material strength

as the test specimen geometry is increased (the so-called size effect). Thus the observed strength values are dependent on specimen size and geometry. Parameter estimates can be computed based on a given specimen geometry, however, the parameter estimates should be transformed and utilized in a component reliability analysis as material-specific parameters. The procedure for transforming parameter estimates for the typical specimen geometries just cited is outlined later in the section entitled "Material Specific MLE Parameters." The user should be aware that the parameters estimated using non-linear regression estimators are material specific parameters. Therefore no transformation is necessary after these parameters have been estimated.



**Figure A.1** Uncensored Sample that possibly demonstrates multiple failure populations.

Advanced ceramics typically contain two or more active flaw distributions (e.g., failures due to inclusions or machining damage) and each will have its own strength distribution parameters. The censoring techniques presented here for the two-parameter Weibull distribution require positive confirmation of multiple flaw distributions, which necessitates fractographic examination to characterize the fracture origin in each specimen. Multiple flaw distributions may also be indicated by a deviation from the linearity of the data from a single Weibull distribution (e.g., Figure A.1). However observations of approximately linear behavior should not be considered a sufficient reason to conclude a single flaw distribution is active. The reader is strongly encouraged to integrate mechanical failure data and fractographic analysis.



**Figure A.2** Censored Sample with multiple failure populations identified.

As was just noted, discrete fracture origins are quite often grouped by flaw distributions. The data for each flaw distribution can also be screened for outliers. An outlying observation is one that deviates significantly from other observations in the sample. The reader should understand that an apparent outlying observation may be an extreme manifestation of the variability in strength. If this is the case, the data point should be retained and treated as any other observation in the failure sample. However, the outlying observation may be the result of a gross deviation from prescribed experimental procedure, or possibly an error in calculating or recording the numerical value of the data point in question. When the experimentalist is clearly aware that either of these situations has occurred, the outlying observation may be discarded, unless the observation (i.e., the strength value) can be corrected in a rational manner. The procedures for dealing with outlying observations are available elsewhere in the literature [2]. For the sake of brevity this discussion omits any discussion on the performance of fractographic analyses, and omits any discussion concerning outlier tests.

## A.2 THE WEIBULL DISTRIBUTION

Experimental data indicates that the continuous random variable representing uniaxial tensile strength of advanced ceramics is asymmetrical about the mean and will assume only positive values. These characteristics rule out the use of the normal distribution (as well as others) and point to the use of the Weibull distribution or a similarly skewed distribution. The three-parameter Weibull probability density function for a continuous random strength variable, denoted as  $\Sigma$ , is given by the expression

$$f_{\Sigma}(\mathbf{s}) = \left(\frac{\mathbf{a}}{\mathbf{b}}\right) \left(\frac{\mathbf{s}-\mathbf{g}}{\mathbf{b}}\right)^{(\mathbf{a}-1)} \exp\left[-\left(\frac{\mathbf{s}-\mathbf{g}}{\mathbf{b}}\right)^{\mathbf{a}}\right] \quad (\text{A.1})$$

for  $\sigma > \gamma$ , and

$$f_{\Sigma}(\mathbf{s}) = 0 \quad (\text{A.2})$$

for  $\sigma \leq \gamma$ . In equation (A.1)  $\alpha$  is the Weibull modulus (or the shape parameter),  $\beta$  is the Weibull scale parameter, and  $\gamma$  is a threshold parameter. The cumulative distribution is given by the expression

$$F_{\Sigma}(\mathbf{s}) = 1 - \exp\left[-\left(\frac{\mathbf{s}-\mathbf{g}}{\mathbf{b}}\right)^{\mathbf{a}}\right] \quad (\text{A.3})$$

for  $\sigma > \gamma$ , and

$$F_{\Sigma}(\mathbf{s}) = 0 \quad (\text{A.4})$$

for  $\sigma \leq \gamma$ .

Often the value of the threshold parameter is taken to be zero. In component design this represents a conservative assumption, and yields the more widely used two-parameter Weibull formulation. Here the expression for the probability density function simplifies to

$$f_{\Sigma}(\mathbf{s}) = \left(\frac{\mathbf{a}}{\mathbf{b}}\right) \left(\frac{\mathbf{s}}{\mathbf{b}}\right)^{(\mathbf{a}-1)} \exp\left[-\left(\frac{\mathbf{s}}{\mathbf{b}}\right)^{\mathbf{a}}\right] \quad (\text{A.5})$$

for  $\sigma > 0$ , and

$$f_{\Sigma}(\mathbf{s}) = 0 \quad (\text{A.6})$$

for  $\mathbf{s} \leq \mathbf{0}$ . The cumulative distribution simplifies to

$$F_{\mathbf{S}}(\mathbf{s}) = 1 - \exp\left[-\left(\frac{\mathbf{s}}{\mathbf{b}}\right)^{\mathbf{a}}\right] \quad (\text{A.7})$$

for  $\mathbf{s} > \mathbf{g}$ , and

$$F_{\mathbf{S}}(\mathbf{s}) = 0 \quad (\text{A.8})$$

for  $\mathbf{s} \leq \mathbf{g}$ . Note that in the ceramics literature when the two parameter Weibull formulation is adopted then "m" is used for the Weibull modulus  $\mathbf{a}$ , and either  $\mathbf{s}_0$  or  $\mathbf{s}_q$  (see the discussion below regarding the difference between  $\mathbf{s}_0$  and  $\mathbf{s}_q$ ) is used for the Weibull scale parameter. The WeibPar program uses "M", "Sig Not", and "Char Str" to notate the parameter estimates in the two parameter formulation, and the program uses "M", "Sig Not", and "Threshold" for the three parameter formulation. In the discussion that follows the  $(\mathbf{a}, \mathbf{b}, \mathbf{g})$  notation is used exclusively and reference is made to the typical notation adopted in the ceramics literature. The reason for this is the tendency to overuse the "S" symbol (e.g.,  $\mathbf{s}_q$ ,  $\mathbf{s}_0$ ,  $\mathbf{s}_i$ -failure observation, and  $\mathbf{s}_t$ -threshold stress, etc.). Throughout this discussion the symbol "S" will imply applied stress.

If the random variable representing uniaxial tensile strength of an advanced ceramic is characterized by a two-parameter Weibull distribution, i.e., the random strength parameter is governed by equations (A.5) and (A.6), then the probability that a uniaxial test specimen fabricated from an advanced ceramic will fail can be expressed by the cumulative distribution function

$$P_f = 1 - \exp\left[-\left(\frac{\mathbf{s}_{max}}{\mathbf{b}_q}\right)^{\mathbf{a}}\right] \quad (\text{A.9})$$

Note that  $\mathbf{s}_{max}$  is the maximum normal stress in the component. The parameter  $\mathbf{b}_q$  is the Weibull characteristic strength which is a location parameter dependent on the type of uniaxial test specimen (e.g., tensile, flexural, or pressurized ring) utilized. Thus  $\mathbf{b}_q$  (which has units of stress) will change with specimen geometry and stress gradients in the test specimen. In the ceramics literature this parameter would correspond to  $\mathbf{s}_q$  for the two parameter formulation. An alternative expression for the probability of failure was derived by Weibull and expressed as

$$P_f = 1 - \exp\left[-\int_V \left(\frac{\mathbf{s}}{\mathbf{b}_q}\right)^{\mathbf{a}} dV\right] \quad (\text{A.10})$$

This integration is performed over all tensile regions of the specimen volume if the strength-controlling flaws are randomly distributed through the volume of the material, or over all tensile regions of the specimen area if flaws are restricted to the specimen surface. The Weibull material scale parameter  $\mathbf{b}_0$  for volume defects has units of [stress  $\cdot$  (volume) $^{1/\mathbf{a}}$ ]. If the strength controlling flaws are restricted to the surface of the specimens in a sample, then the Weibull material scale parameter has units of [stress  $\cdot$  (area) $^{1/\mathbf{a}}$ ]. This parameter corresponds to  $\mathbf{s}_0$  in the

ceramics literature for the two parameter formulation of the Weibull distribution. From a computational standpoint an estimate for  $\mathbf{b}_q$  is obtained from the failure data. This value is converted to an equivalent  $\mathbf{b}_0$  value. To perform this transformation equations (A.9) and (A.10) can be equated for the test specimen geometry. The resulting expression yields a relationship between  $\mathbf{b}_0$  and  $\mathbf{b}_q$  for that specific specimen geometry. Expressions for the tensile specimen geometry and flexural specimen geometry appear later in this appendix.

### A.3 MAXIMUM LIKELIHOOD ESTIMATORS

The maximum likelihood technique has certain advantages, especially when parameters must be determined from censored failure populations. When a sample of test specimens yields two or more distinct flaw distributions, the sample is said to contain censored data. The methods described in this discussion include censoring techniques that apply to multiple concurrent flaw distributions. A concurrent flaw distribution is found in a homogeneous material if every test specimen fabricated from that material contains representative flaws from each independent flaw population. Within a given specimen all flaw populations are present concurrently, and the flaw distributions are competing with each other to cause failure. Thus this term is synonymous with “competing flaw distributions.” The methods for parameter estimation presented in this discussion are not applicable to data sets that contain exclusive or compound multiple flaw distributions (see [3] for a more detailed discussion on this topic). A simple example of a compound flaw distribution is where every specimen contains the flaw distribution A, while some fraction of the specimens also contains a second independent flaw distribution B. An exclusive flaw distribution is a type of multiple flaw distribution created by mixing and randomizing specimens from two or more versions of a material where each version contains a different single flaw population. Thus, each specimen contains flaws exclusively from a single distribution, but the total data set reflects more than one type of strength-controlling flaw.

The parameter estimates obtained using the maximum likelihood technique are unique (for a two-parameter Weibull distribution), and as the size of the sample increases, the estimates statistically approach the true values of the population. Let  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$  represent the ultimate strength (a random variable) of the ceramic test specimens in a given sample. It is assumed that the ultimate strength is characterized by the two-parameter Weibull distribution. The likelihood function associated with this sample is the joint density of the  $N$  random variables, and thus is a function of the unknown Weibull distribution parameters  $(\mathbf{a}, \mathbf{b})$ . The likelihood function for a censored sample under these assumptions is given by the expression

$$\mathcal{L} = \left\{ \prod_{i=1}^r \left( \frac{\tilde{\mathbf{a}}}{\tilde{\mathbf{b}}_q} \right) \left( \frac{\mathbf{s}_i}{\tilde{\mathbf{b}}_q} \right)^{\tilde{\mathbf{a}}-1} \exp \left[ - \left( \frac{\mathbf{s}_i}{\tilde{\mathbf{b}}_q} \right)^{\tilde{\mathbf{a}}} \right] \right\} \left\{ \prod_{j=r+1}^N \exp \left[ - \left( \frac{\mathbf{s}_j}{\tilde{\mathbf{b}}_q} \right)^{\tilde{\mathbf{a}}} \right] \right\} \quad (\text{A.11})$$

This expression can be applied to a sample where two or more concurrent flaw distributions have been identified from fractographic inspection. For the purpose of discussion consider different distributions identified as flaw types A, B, C, etc. When equation (A.11) is used to estimate the parameters associated with the type-A flaw distribution, then  $r$  is the number of specimens where type-A flaws were found at the fracture origin, and  $i$  is the associated index in the first summation. The second summation is carried out for all other specimens not failing from type-A flaws (i.e., type-B flaws, type-C flaws, etc.). Therefore the sum is carried out from ( $j=r+1$ ) to  $N$  (the total number of specimens) where  $j$  is the index in the second summation. Accordingly,  $\mathbf{s}_i$  is the  $i^{\text{th}}$  failure stress for specimens associated with type-A flaws. In a similar fashion  $\sigma_j$  is associated with the other flaw types present. The likelihood function for the two-parameter Weibull distribution for a single flaw population is defined by the expression

$$\mathcal{L} = \prod_{i=1}^N \left( \frac{\tilde{\mathbf{a}}}{\tilde{\mathbf{b}}_q} \right) \left( \frac{\mathbf{s}_i}{\tilde{\mathbf{b}}_q} \right)^{\tilde{\mathbf{a}}-1} \exp \left[ - \left( \frac{\mathbf{s}_i}{\tilde{\mathbf{b}}_q} \right)^{\tilde{\mathbf{a}}} \right] \quad (\text{A.12})$$

where  $r$  was taken equal to  $N$  in equation (A.11). The parameter estimates for the Weibull modulus and the characteristic strength are determined by taking the partial derivatives of the logarithm of the likelihood function with respect to  $\tilde{\mathbf{a}}$  as well as  $\tilde{\mathbf{b}}_q$  and equating the resulting expressions to zero. The system of equations obtained by differentiating the log likelihood function for a censored sample is given by [4]

$$\frac{\sum_{i=1}^N (\mathbf{s}_i)^{\tilde{\mathbf{a}}} \ln (\mathbf{s}_i)}{\sum_{i=1}^N (\mathbf{s}_i)^{\tilde{\mathbf{a}}}} - \frac{1}{N} \sum_{i=1}^N \ln (\mathbf{s}_i) - \frac{1}{\tilde{\mathbf{a}}} = 0 \quad (\text{A.13})$$

and

$$\tilde{\mathbf{b}}_q = \left[ \left( \sum_{i=1}^N (\mathbf{s}_i)^{\tilde{\mathbf{a}}} \right) \frac{1}{N} \right]^{\frac{1}{\tilde{\mathbf{a}}}} \quad (\text{A.14})$$

Once again  $r$  is the number of failed specimens from a particular group of a censored sample. Thus when a sample does not require censoring,  $r$  is replaced by  $N$  in equations (A.13) and (A.14). The WeibPar program numerically solves equation (A.13) first since a closed form solution for  $\tilde{\mathbf{a}}$  can not be obtained from this expression. Once  $\tilde{\mathbf{a}}$  is determined this value is inserted into equation (A.14) and  $\tilde{\mathbf{b}}_q$  is calculated directly.

#### A.4 MATERIAL SPECIFIC MLE PARAMETERS

Relationships between the estimate of the Weibull characteristic strength and the Weibull material scale parameter for any specimen configuration can be derived by equating the expressions given by equations (A.9) and (A.10) with the modifications that follow. Begin by performing the integration given in equation (A.10) such that

$$P_f = 1 - \exp\left[-kV\left(\frac{s_{max}}{b}\right)^a\right] \quad (\text{A.15})$$

Here  $k$  is a dimensionless constant that accounts for specimen geometry and stress gradients [3]. Note that in general,  $k$  is a function of the estimated Weibull modulus  $\tilde{a}$ , and is always less than or equal to unity. The product  $kV$  is often referred to as the effective volume (with the designation  $V_E$ ). The effective volume can be interpreted as the size of an equivalent uniaxial tensile specimen that has the same risk of rupture as the test specimen or component. As the term implies, the product represents the volume of material subject to a uniform uniaxial tensile stress. Setting equations (A.9) and (A.10) equal to one another yields the following expression

$$\left(\tilde{b}_o\right)_V = (kV)^{1/\tilde{a}_V} \left(\tilde{b}_q\right)_V \quad (\text{A.16})$$

where the subscript  $V$  attached to the parameter estimates denotes a volume integration. Thus for an arbitrary test specimen, the experimentalist evaluates the integral identified in equation (A.10) for the effective volume ( $kV$ ), and utilizes equation (A.16) to obtain the estimated Weibull material scale parameter  $\tilde{b}_o$ . A similar procedure can be adopted when fracture origins are spatially distributed at the surface of the test specimen.

As an example, the following equation defines the relationship between the parameters for tensile specimens

$$\left(\tilde{b}_o\right)_V = (V)^{1/\tilde{a}_V} \left(\tilde{b}_q\right)_V \quad (\text{A.17})$$

where  $V$  is the volume of the uniform gage section of the tensile specimen, and the fracture origins are spatially distributed within this volume. For a tensile specimen in which fracture origins are spatially distributed at the surface of the specimens tested,

$$\left(\tilde{b}_o\right)_A = (A)^{1/\tilde{a}_A} \left(\tilde{b}_q\right)_A \quad (\text{A.18})$$

where  $A$  is the surface area of the uniform gage section.

For flexural specimen geometries, the relationships become more complex. The following relationship is based on the geometry of a flexural specimen found in Figure A.3. For fracture origins spatially distributed within both the volume of a flexural specimen and the outer load span

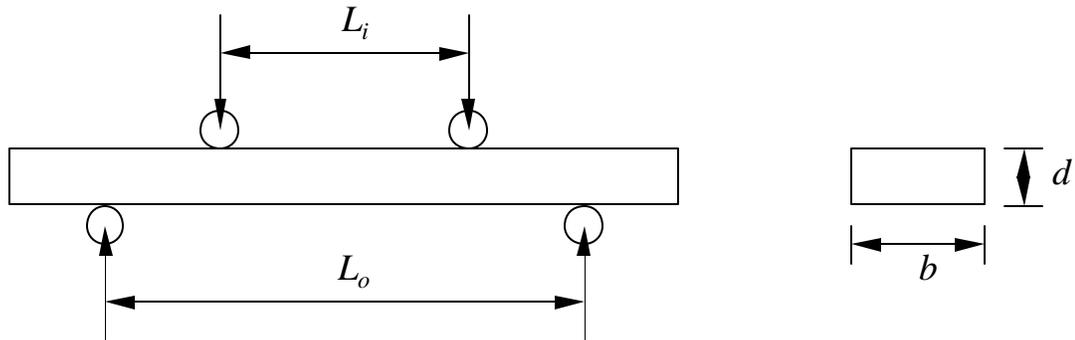
$$(\tilde{b}_o)_V = (\tilde{b}_q)_V \left\{ \frac{V \left[ \left( \frac{L_i}{L_o} \right) \tilde{a}_V + I \right]}{2(\tilde{a}_V + I)^2} \right\}^{\frac{1}{\tilde{a}_V}} \quad (\text{A.19})$$

where  $L_i$  is the length of the inner load span,  $L_o$  is the length of the outer load span, and  $V$  is the volume of the gage section defined by the expression

$$V = b d L_o \quad (\text{A.20})$$

The dimensions  $b$  and  $d$  are identified in Figure A.3. For fracture origins spatially distributed at the surface of a flexural specimen and within the outer load span,

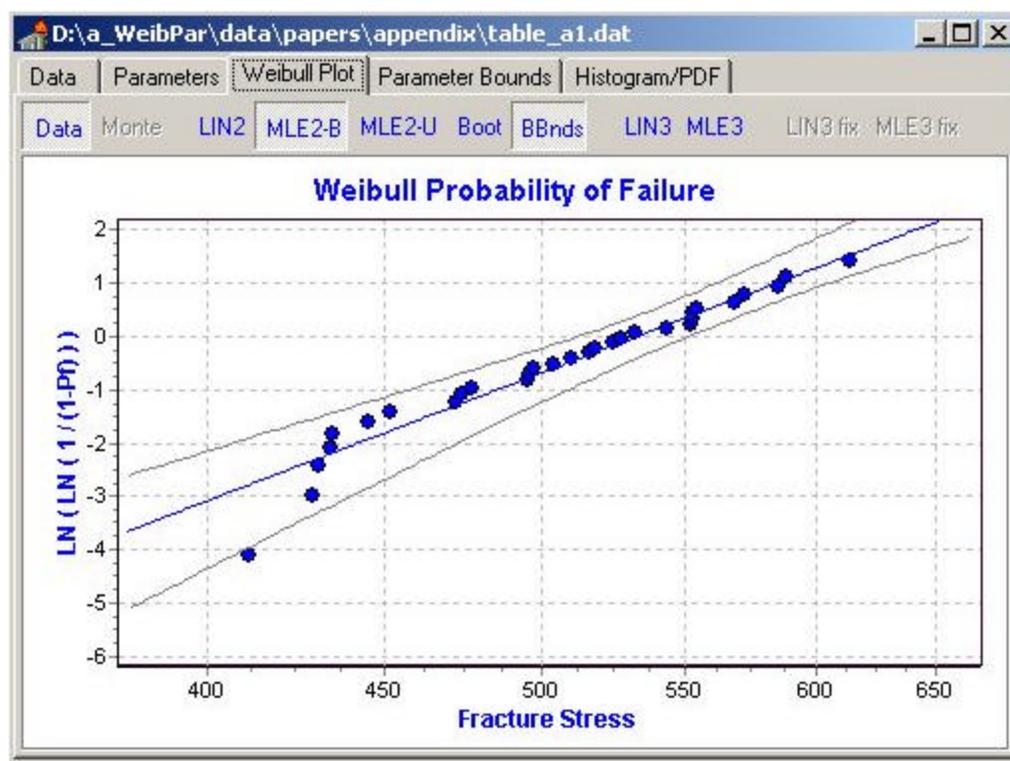
$$(\tilde{b}_o)_A = (\tilde{b}_q)_A \left\{ L_o \left[ \left( \frac{d}{\tilde{a} + I} \right) + b \right] \left[ \frac{\left( \frac{L_i}{L_o} \right) \tilde{a}_A + I}{\tilde{a}_A + I} \right] \right\}^{\frac{1}{\tilde{a}_A}} \quad (\text{A.21})$$



**Figure A.3** Geometry for a flexural test specimen.

**Table A.1** Silicon Nitride Fracture Stress Data Utilized in Maximum Likelihood Estimation

Specimen	Stress	Specimen	Stress	Specimen	Stress
1	411.0 MPa	11	495.0 MPa	21	543.0 MPa
2	429.0 MPa	12	496.0 MPa	22	552.0 MPa
3	431.0 MPa	13	497.0 MPa	23	553.0 MPa
4	434.0 MPa	14	504.0 MPa	24	553.0 MPa
5	435.0 MPa	15	510.0 MPa	25	554.0 MPa
6	445.0 MPa	16	516.0 MPa	26	568.0 MPa
7	452.0 MPa	17	518.0 MPa	27	572.0 MPa
8	472.0 MPa	18	524.0 MPa	28	585.0 MPa
9	474.0 MPa	19	527.0 MPa	29	588.0 MPa
10	477.0 MPa	20	532.0 MPa	30	614.0 MPa



**Figure A.4** Silicon nitride failure data (see Table A.1) and the probability of failure curve (blue line) based on estimated values Maximum likelihood estimators of the Weibull parameters. The 95% confidence bounds (black curves) based on the Bootstrap technique are also shown.

In order to demonstrate how the previous discussion is utilized, consider the failure data in Table A.1. The data represent the maximum stress at failure for bend specimens (four-point) fabricated from HIP'ed (hot isostatically pressed) silicon nitride [5]. The solution of equation (A.13) requires an iterative numerical scheme. Using the WeibPar program a parameter estimate for the Weibull modulus of  $\tilde{\alpha} = 10.75$  was obtained. Subsequent solution of equation (A.14) yields a value of  $\tilde{b}_q = 533$  MPa. These values for the Weibull parameter estimates were generated by assuming a unimodal failure sample with no censoring (i.e.,  $r = N$ ). Figure A.4 depicts the individual failure data and a curve based on the estimated values of the parameters. Next, assuming that the failure origins were distributed at the surface of the specimens and then inserting the estimated values of  $\tilde{\alpha}$  and  $\tilde{b}_q$  into equation (A.21) along with the specimen geometry (i.e.,  $L_o = 40$  mm,  $L_i = 20$  mm,  $d = 3$  mm, and  $b = 4$  mm) yields  $(\tilde{b}_0)_A = 811.6$  MPa  $\times$  (m<sup>2</sup>)<sup>1/10.75</sup>. Alternatively, if one were to assume that the failure origins were volume distributed, then the solution of equation (A.19) yields  $(\tilde{b}_0)_V = 666.3$  MPa  $\times$  (m<sup>3</sup>)<sup>1/10.75</sup>. The different values obtained from assuming surface and volume fracture origins underscore the necessity of conducting a fractographic analysis.

## A.5 UNBIASING FACTORS AND CONFIDENCE INTERVALS FOR MAXIMUM LIKELIHOOD ESTIMATES

If all failures from a group of observations originate from a single flaw distribution an unbiased estimate of the Weibull modulus can be computed. Procedures for bias correction and computing confidence intervals in the presence of multiple active flaw populations are not well developed at this time. In addition, unbiasing factors and parameters utilized to establish confidence bounds are only available for likelihood estimates of the two-parameter Weibull distribution. Statistical bias can be defined numerically in the following manner. Consider distributions of point estimates generated numerically using Monte Carlo techniques. These distributions are obtained by numerous computer generated samples and the resulting point estimates are ranked for each sample size. If the mean value of the ranked data is equal to the expected value of the true parameter for a given sample size, the estimator is said to be unbiased.

If an estimator yields biased results the value of the individual estimates can be corrected if the estimators are invariant (see Thoman et al. [6] for a proof of invariance for the two-parameter maximum likelihood estimators presented earlier). The bias associated with the estimate of the characteristic strength is minimal (<0.3% for 20 test specimens, as opposed to  $\cong 7\%$  bias for  $\tilde{\mathbf{a}}$  with the same number of specimens), and is usually ignored. However, the WeibPar program enables allows for the unbiasing of the Weibull modulus and the Weibull characteristic strength. The user should also keep in mind that statistical bias associated with the maximum likelihood estimators presented here can always be reduced by increasing the sample size.

The amount of deviation between the biased estimate and the expected value of the true parameter can be quantified either as a percent difference or with unbiasing factors. In keeping with the accepted practice in the open literature, statistical bias is quantified in the WeibPar program through the use of unbiasing factors (denoted as  $UF$ ). Unbiasing factors (as well as the ratios used to compute upper and lower bounds for a confidence interval) are obtained from the ranked distributions of point estimates mentioned above. In the WeibPar program these unbiasing factors are located in "look-up" tables that are accessed directly by the program. The program computes unbiased values of  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{b}}_q$  directly, i.e., this calculation is transparent to the user. The user should note that the "look-up" tables of unbiasing factors for  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{b}}_q$  in the WeibPar program are far more extensive than the tables published in reference [8].

As an example of computing unbiased estimates of the Weibull modulus consider the same unimodal failure sample presented in Table A.1. The sample contained 30 specimens and the biased estimate of the Weibull modulus was determined to be  $\tilde{\mathbf{a}} = 10.75$ . The unbiasing factor corresponding to this sample size is  $UF = 0.953$  (obtained from the "look-up" tables). Thus, the unbiased estimate of the Weibull modulus is given as

$$\begin{aligned}\tilde{\mathbf{a}}_u &= \tilde{\mathbf{a}} \times UF \\ &= (10.75)(0.953) \\ &= 10.24\end{aligned}\tag{A.22}$$

Confidence intervals quantify the uncertainty associated with a point estimate of a population parameter. The size of a confidence interval for maximum likelihood estimates of both Weibull parameters will diminish with increasing sample size. The values used to construct a confidence interval are based on percentile distributions obtained by the Monte Carlo simulations mentioned earlier in this section. For example, the 90% confidence interval for the Weibull modulus is obtained from the 5 and 95 percentile distributions of the ratio of  $\tilde{\mathbf{a}}$  to the true population value  $\mathbf{a}$ . The ratios ( $\tilde{\mathbf{a}}/\mathbf{a}$ ) necessary to construct the 90% confidence interval can be found in Table 2 of reference [7]. However, reference [7] limits the user to just the 90% confidence interval. The WeibPar program contains the values needed to compute the lower confidence bounds associated with the 2.5%, 5%, 7.5%, 10%, 12.5%, 15%, 20%, 25% percentile distributions. Similarly, the WeibPar program contains the values needed to compute the upper confidence bound associated with the 75%, 80%, 85%, 87.5%, 90%, 92.5%, 95%, 97.5% percentile distributions. Thus by carefully selecting the upper and lower confidence bounds the user can construct a number of different confidence intervals. Finally the user should keep in mind that the biased estimate of the Weibull modulus must be used to construct the confidence bounds.

Confidence intervals can also be constructed for the estimated Weibull characteristic strength. However, the percentile distributions needed to construct the intervals do not involve the same normalized ratios or the same tables as those used for the Weibull modulus. Define the function

$$t = \tilde{\mathbf{a}} \ln \left( \frac{\tilde{\mathbf{b}}_q}{\mathbf{b}_q} \right) \quad (\text{A.23})$$

The 90% confidence interval for the characteristic strength is obtained from the 5 and 95 percentile distributions of  $t$ . For the point estimate of the characteristic strength, these percentile distributions can be found in Table 3 of reference [7]. However, reference [7] limits the user to just the 90% confidence bounds. The WeibPar program contains the values needed to compute the lower confidence bounds associated with the 2.5%, 5%, 7.5%, 10%, 12.5%, 15%, 20%, 25% percentile distributions. Similarly, the WeibPar program contains the values needed to compute the upper confidence bound associated with the 75%, 80%, 85%, 87.5%, 90%, 92.5%, 95%, 97.5% percentile distributions. Thus by carefully selecting the upper and lower confidence bounds the user can construct a number of different confidence intervals. Note that the biased estimate of the Weibull modulus must also be used here. Again, this procedure is not appropriate for censored statistics. In addition, the reader is cautioned that equation (A.23) can not be utilized in developing confidence bounds on  $\tilde{\mathbf{b}}_0$ . Therefore the confidence bounds on  $\tilde{\mathbf{b}}_q$  should not be converted through the use of equations (A.9) and (A.10).

The upper bound for the 90% confidence interval associated with  $\tilde{\mathbf{a}}$  for the sample presented in Table A.1 is given by

$$\begin{aligned}
 \tilde{\mathbf{a}}_{upper} &= \tilde{\mathbf{a}}/q_{0.95} \\
 &= (10.75)/(0.822) \\
 &= 13.08
 \end{aligned}
 \tag{A.24}$$

where  $q_{0.95}$  is obtained from Table 2 of reference [7], or the appropriate "look-up" table associated with the WeibPar program, for a sample size of 30 failed specimens. The lower bound is

$$\begin{aligned}
 \tilde{\mathbf{a}}_{lower} &= \tilde{\mathbf{a}}/q_{0.05} \\
 &= (10.75)/(1.335) \\
 &= 8.05
 \end{aligned}
 \tag{A.25}$$

where  $q_{0.05}$  is also obtained from Table 2 of reference [7], or the appropriate "look-up" table associated with the WeibPar program.

Similarly, the upper bound for the 90% confidence interval associated with  $\tilde{\mathbf{b}}_q$  is

$$\begin{aligned}
 \tilde{\mathbf{b}}_{q\ upper} &= \tilde{\mathbf{b}}_q \exp\left(\frac{-t_{0.05}}{\tilde{\mathbf{a}}}\right) \\
 &= (533) \exp\left(\frac{0.332}{10.75}\right) \\
 &= 550\ MPa
 \end{aligned}
 \tag{A.26}$$

where  $t_{0.05}$  is obtained from Table 3 of reference [7], or the appropriate "look-up" table associated with the WeibPar program, for a sample size of 30 failed specimens. The lower bound on  $\tilde{\mathbf{b}}_q$  is

$$\begin{aligned}
 \tilde{\mathbf{b}}_{q\ lower} &= \tilde{\mathbf{b}}_q \exp\left(\frac{-t_{0.95}}{\tilde{\mathbf{a}}}\right) \\
 &= (533) \exp\left(\frac{-0.335}{10.75}\right) \\
 &= 517\ MPa
 \end{aligned}
 \tag{A.27}$$

where  $t_{0.95}$  is also obtained from Table 3 of reference [7], or the appropriate "look-up" table associated with the WeibPar program. Thus it can be stated with 90% confidence that the estimate of the Weibull modulus for this material is bounded such that  $8.05 \leq \tilde{\mathbf{a}} \leq 13.08$ . Similarly, it can be stated that the estimate of the characteristic strength is bounded such that  $517 \leq \tilde{\mathbf{b}}_q \leq 550$ . The size of these bounds depend directly on the sample size. If the bounds in this particular case were unacceptable, then the sample size should be increased. The size of the confidence intervals addresses the question of how many samples are sufficient.

## A.6 NON-LINEAR REGRESSION ESTIMATORS FOR A THREE-PARAMETER WEIBULL DISTRIBUTION

To date, most reliability analyses performed on structural components fabricated from ceramic materials have utilized the two-parameter form of the Weibull distribution. The use of a two-parameter Weibull distribution to characterize the random nature of material strength implies a non-zero probability of failure for the full range of applied stress. This represents a conservative design assumption when analyzing structural components. The three-parameter form of the Weibull distribution was presented earlier in equations (A.1) and (A.2). The additional parameter is a threshold stress ( $g$ ) that allows for zero probability of failure when the applied stress is at or below the threshold value. Certain monolithic ceramics have exhibited threshold behavior. The reader is directed to an extensive data base assembled by Quinn [8], the silicon nitride data in Foley et al. [9], as well as data (with supporting fractography) presented by Chao and Shetty [10] that was analyzed later in Duffy et al. [1].

When strength data indicates the existence of a threshold stress, a three-parameter Weibull distribution should be employed in the stochastic failure analysis of structural components. By employing the concept of a threshold stress, an engineer can effectively tailor the design of a component to optimize structural reliability. To illustrate the approach Duffy et al. [1] embedded the three-parameter Weibull distribution in a reliability model that utilized the principle of independent action (PIA). Analysis of a space shuttle main engine (SSME) turbo-pump blade predicted a substantial improvement in component reliability when the three-parameter Weibull distribution was utilized in place of the two-parameter Weibull distribution. Note that the three-parameter form of the Weibull distribution can easily be extended to Batdorf's [11,12] model, reliability models proposed for ceramic composites (see Duffy et al. [13], or Thomas and Wetherhold [14]), as well as the interactive and noninteractive reliability models presented earlier.

The non-linear regression method proposed by Margetson and Cooper [15] is highlighted. Coding for the non-linear regression estimators have been formulated for two basic test configurations: the four-point bend specimen and the uniaxial test specimen. However, these estimators maintain certain disadvantages relative to bias and invariance, and these issues were explored numerically in Duffy et al. [1]. The Monte Carlo simulations in Duffy et al. [1] demonstrated that the functions proposed by Margetson and Cooper [15] are neither invariant nor unbiased. However, they are asymptotically well-behaved in that bias decreases and confidence intervals contract as the sample size increases. Thus, even though bias and confidence bounds may never be quantified using these non-linear regression technique, the user is guaranteed that estimated values improve as the sample size is increased.

Regression analysis postulates a relationship between two variables. In an experiment typically one variable can be controlled (the independent variable) while the response variable (or dependent variable) is not. In simple failure experiments the material dictates the strength at failure, indicating that the failure stress is the response variable. The ranked probability of failure ( $P_i$ ) can be controlled by the experimentalist, since it is functionally dependent on the sample size ( $N$ ). After arranging the observed failure stresses ( $s_1, s_2, s_3, \dots, s_N$ ) in ascending order, and specifying

$$P_i = \frac{(i-0.5)}{N} \quad (\text{A.28})$$

then clearly the ranked probability of failure for a given stress level can be influenced by increasing or decreasing the sample size. The procedure proposed by Margetson and Cooper [15] adopts this philosophy. They assume that the specimen failure stress is the dependent variable, and the associated ranked probability of failure becomes the independent variable.

Using the three parameter version of equation (A.10), an expression can be obtained relating the ranked probability of failure ( $P_i$ ) to an estimate of the failure strength ( $\mathbf{s}_i$ ). Assuming uniaxial stress conditions in a test specimen with a unit volume, equation (A.10) yields

$$\tilde{\mathbf{s}}_i = \tilde{\mathbf{g}} + \tilde{\mathbf{b}} \left[ \ln \left( \frac{1}{1 - P_i} \right) \right]^{1/\tilde{\mathbf{a}}} \quad (\text{A.29})$$

where  $\tilde{\mathbf{a}}$ ,  $\tilde{\mathbf{b}}$  and  $\tilde{\mathbf{g}}$  are estimates of the shape parameter ( $\mathbf{a}$ ), the scale parameter ( $\mathbf{b}$ ), and threshold parameter ( $\mathbf{g}$ ), respectively. Expressions for the evaluation of these parameters for a test specimen subjected to pure bending are found in Duffy et al. [1]. Defining the residual as

$$\mathbf{d}_i = \tilde{\mathbf{s}}_i - \mathbf{s}_i \quad (\text{A.30})$$

where  $\mathbf{s}_i$  is the  $i^{\text{th}}$  ranked failure stress obtained from actual test data, then the sum of the squared residuals is expressed as

$$\sum_{i=1}^N (\mathbf{d}_i)^2 = \sum_{i=1}^N (\tilde{\mathbf{g}} + \tilde{\mathbf{b}} (W_i)^{1/\tilde{\mathbf{a}}} - \mathbf{s}_i)^2 \quad (\text{A.31})$$

Here the notation of Margetson and Cooper [15] has been adopted where

$$W_i = \ln \left( \frac{1}{1 - P_i} \right) \quad (\text{A.32})$$

Note that the forms of  $\mathbf{s}_i$  and  $W$  change with specimen geometry (see the previous discussion relating to the four-point bend specimen geometry). It should be apparent that the objective of this method is to obtain parameter estimates that minimize the sum of the squared residuals. Setting the partial derivatives of the sum of the squared residuals with respect to  $\tilde{\mathbf{a}}$ ,  $\tilde{\mathbf{b}}$  and  $\tilde{\mathbf{g}}$  equal to zero yields the following three expressions

$$\tilde{\mathbf{b}} = \frac{N \left[ \sum_{i=1}^N \mathbf{s}_i (W_i)^{1/\tilde{\mathbf{a}}} \right] - \left[ \sum_{i=1}^N \mathbf{s}_i \right] \left[ \sum_{i=1}^N (W_i)^{1/\tilde{\mathbf{a}}} \right]}{N \sum_{i=1}^N (W_i)^{2/\tilde{\mathbf{a}}} - \left[ \sum_{i=1}^N (W_i)^{1/\tilde{\mathbf{a}}} \right] \left[ \sum_{i=1}^N (W_i)^{1/\tilde{\mathbf{a}}} \right]} \quad (\text{A.33})$$

$$\tilde{\mathbf{g}} = \frac{\left[ \sum_{i=1}^N \mathbf{s}_i \right] \left[ \sum_{i=1}^N (W_i)^{2/\tilde{\mathbf{a}}} \right] - \left[ \sum_{i=1}^N \mathbf{s}_i (W_i)^{1/\tilde{\mathbf{a}}} \right] \left[ \sum_{i=1}^N (W_i)^{1/\tilde{\mathbf{a}}} \right]}{N \sum_{i=1}^N (W_i)^{2/\tilde{\mathbf{a}}} - \left[ \sum_{i=1}^N (W_i)^{1/\tilde{\mathbf{a}}} \right] \left[ \sum_{i=1}^N (W_i)^{1/\tilde{\mathbf{a}}} \right]} \quad (\text{A.34})$$

and

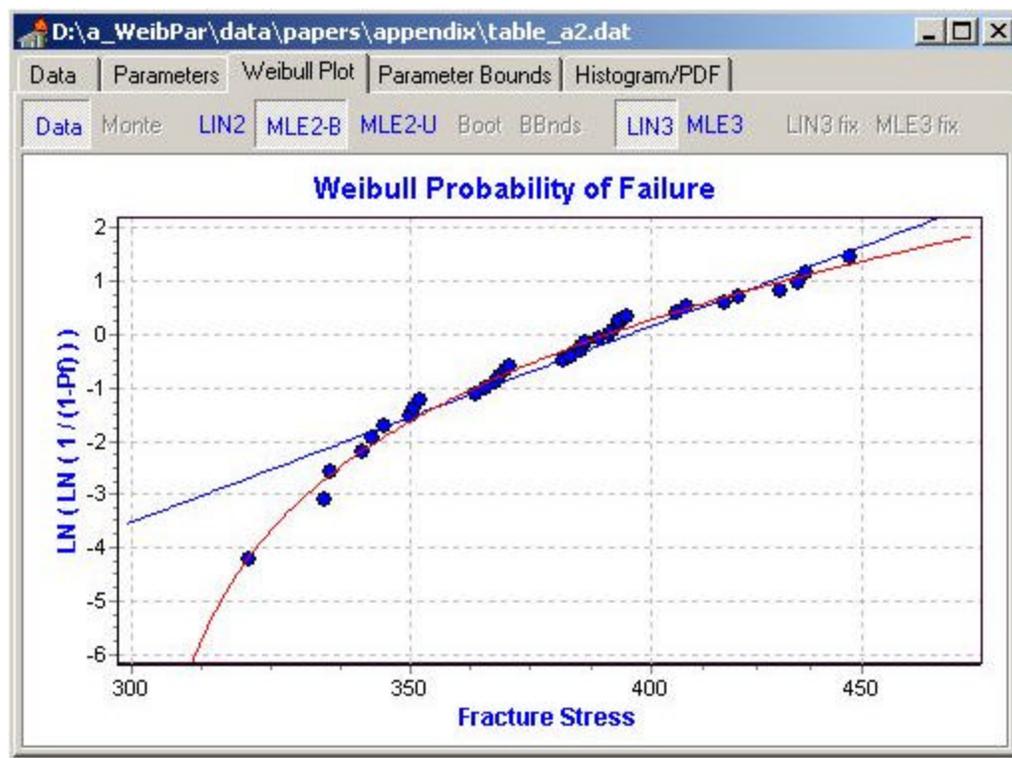
$$\left| \sum_{i=1}^N \mathbf{s}_i (W_i)^{1/\tilde{\mathbf{a}}} \ln(W_i) - \tilde{\mathbf{g}} \sum_{i=1}^N \mathbf{s}_i (W_i)^{1/\tilde{\mathbf{a}}} \ln(W_i) - \tilde{\mathbf{b}} \sum_{i=1}^N \mathbf{s}_i (W_i)^{2/\tilde{\mathbf{a}}} \ln(W_i) \right| \leq \mathbf{k}_{conv} \quad (\text{A.35})$$

in terms of the parameter estimates. The solution of this system of equations is iterative, where the third expression is used to check convergence at each iteration. The initial solution vector for this system is determined after assuming  $\tilde{\mathbf{a}}=1$ . Then  $\tilde{\mathbf{b}}$  is computed from equation (A.33) and  $\tilde{\mathbf{g}}$  is calculated from equation (A.34). The values of these parameter estimates are then inserted into equation (A.35) to determine if the convergence criterion is satisfied to within some predetermined tolerance ( $\mathbf{k}_{conv}$ ). If this expression is not satisfied,  $\tilde{\mathbf{a}}$  is updated and a new iteration is conducted. This procedure continues until a set of parameter estimates are determined that satisfy equation (A.35).

The estimators perform reasonably well in comparison to estimates of the two-parameter Weibull distribution for the alumina data found in Table A.2. Figure A.5 is a plot of probability of failure versus failure stress for this data. The straight line represents the two parameter maximum likelihood fit to the data where  $\tilde{\mathbf{a}} = 12.7$ ,  $\tilde{\mathbf{b}} = 395$  ( $\tilde{\mathbf{g}} \equiv 0$ ). The non-linear curve represents the three parameter linear regression fit to the data where  $\tilde{\mathbf{a}} = 2.71$ ,  $\tilde{\mathbf{b}} = 89.7$ , and  $\tilde{\mathbf{g}} = 301$ . Note that the three-parameter distribution appears more efficient in predicting the failure data in the high reliability region of the graph. This is a qualitative assessment. Goodness-of-fit statistics such as the Kolmogorov-Smirnov statistic, the Anderson-Darling statistic, and likelihood ratios provide quantitative measures to establish which form of the Weibull distribution would best fit the experimental data. These statistics are utilized in conjunction with hypothesis testing to assess the significance level at which the null hypothesis can be rejected. Comparisons can then be made based on the value of the significance level.

**Table A.2** Alumina Fracture Stress Data Utilized in Nonlinear Regression Estimation

Specimen	Stress	Specimen	Stress	Specimen	Stress
1	320 MPa	13	368 MPa	24	393 MPa
2	334 MPa	14	369 MPa	25	393 MPa
3	335 MPa	15	370 MPa	26	395 MPa
4	341 MPa	16	381 MPa	27	406 MPa
5	343 MPa	17	383 MPa	28	408 MPa
6	345 MPa	18	385 MPa	29	417 MPa
7	350 MPa	19	385 MPa	30	420 MPa
8	351 MPa	20	386 MPa	31	430 MPa
9	352 MPa	21	389 MPa	32	434 MPa
10	363 MPa	22	391 MPa	33	436 MPa
11	365 MPa	23	392 MPa	34	447 MPa
12	367 MPa				

**Figure A.5** Alumina failure data (see Table A.2) and probability of failure curves based on estimated parameters for the two- and three-parameter Weibull distributions.

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